Journal of Engineering Physics and Thermophysics, Vol. 80, No. 5, 2007

AN EXACT SOLUTION TO THE UNSTEADY FREE-CONVECTION BOUNDARY-LAYER FLOW PAST AN IMPULSIVELY STARTED VERTICAL SURFACE WITH NEWTONIAN HEATING

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UDC 536.25

An exact solution of the unsteady free-convection boundary-layer flow of an incompressible fluid past an infinite vertical plate with the flow generated by Newtonian heating and impulsive motion of the plate is presented. The resulting governing equations are nondimensionalized and their solutions are obtained in a closed form with the help of the Laplace transform technique. A parametric study of the roles of all involved parameters is conducted and a representative set of numerical results for the velocity, temperature, and skin friction is illustrated graphically. The physical aspects of the problem are discussed.

Introduction. The study of unsteady boundary layer is useful in several physical problems such as flow over a helicopter in translation motion, flow over the blades of turbines and compressors, flow over the aerodynamic surfaces of vehicles in manned flight, etc. The unsteadiness in the flow field is caused by either time dependent or impulsive motion of an external stream (or of the body surface). When the fluid motion over a body is impulsive, the inviscid flow in this range is developed instantaneously but the viscous layer near the body is developed slowly and becomes a fully developed steady-state viscous flow after a certain instant of time. Unsteady laminar free convection past an infinite vertical plate for the Prandtl number Pr = 1.0 in the case of a step change in the wall temperature with time was considered by Illingworth [1]; for Pr ≠ 1.0 he derived the solution in integral form. Siegel [2] studied an unsteady free-convection flow past a semi-infinite vertical plate with a step change in the wall temperature or surface heat flux by the momentum integral method. He was the first to point out that the initial behavior of the temperature and velocity fields for a semi-infinite vertical flat plate is the same as for a double-infinite vertical flat plate, i.e., the temperature field is the same as in the solution of an unsteady one-dimensional heat-conduction problem. Soundalgekar [3] was the first to present an exact solution to the flow of a viscous incompressible fluid past an impulsively started infinite vertical plate by the Laplace transform technique. An excellent review of existing theoretical and experimental works can be found in the books by Stuart [4], Telionis [5], and Pop [6]. Theoretical studies on the laminar natural convection heat transfer from a vertical plate continue to attract attention due to their industrial and technological applications. Martynenko et al. [7] investigated the laminar free convection from a vertical plate maintained at a constant temperature which is equal to the temperature of the surrounding stationary fluid. Perdikis [8] studied free-convection effects on flow past a moving plate. Camargo et al. [9] presented a numerical study of natural convective cooling. Recently Raptis and Perdikis [10] studied the free-convection flow of water near a moving plate. The unsteady free-convection flow with heat flux and accelerated boundary motion was investigated by Chandran et al. [11]. Das et al. [12] analyzed the problem of flow with periodic temperature variation, and Muthucumaraswamy [13] considered natural convection with a variable surface heat flux. Recently, Chandran [14] studied natural convection with a ramped wall temperature.

In all the studies cited above, the flow is driven either by a prescribed surface temperature or by a prescribed surface heat flux. Here, a somewhat different driving mechanism for unsteady free convection along a vertical surface is considered, where it is assumed that the flow is also set up by Newtonian heating from the surface. Heat-transfer characteristics are dependent on the thermal boundary conditions. In general, there are four common heating processes representing the wall-to-ambient temperature distribution, prescribed surface heat flux distribution, and conjugate condi-

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Published in Inzhenerno-Fizicheskii Zhurnal, Vol. 80, No. 5, pp. 102–107, September–October, 2007. Original article submitted February 28, 2006; revision submitted November 22, 2006.

tions, where heat transfer through a bounding surface of finite thickness and finite heat capacity is specified. The interface temperature is not known a priori but depends on the intrinsic properties of the system, namely, the thermal conductivities of the fluid and solid. In Newtonian heating, the rate of heat transfer from the bounding surface with a finite heat capacity is proportional to the local surface temperature, and it is usually termed conjugate convective flow. This situation occurs in many important engineering devices, for example:

(a) in heat exchangers, where conduction in the solid tube wall is greatly influenced by convection in the fluid flowing past it;

(b) in conjugate heat transfer around fins, where conduction within the fin and convection in the fluid surrounding it must be simultaneously analyzed in order to obtain the vital design information;

(c) in convective flow setups, where the bounding surfaces absorb the heat of solar radiation.

Therefore we conclude that the conventional assumption of the absence of interrelation between coupled conduction-convection effects is not always realistic, and this interrelation must be considered when evaluating the conjugate heat transfer processes in many practical engineering applications.

The Newtonian heating condition has only recently been used in studying convective heat transfer. Merkin [15] was the first to consider the free-convection boundary layer over a vertical flat plate immersed in a viscous fluid, whereas the authors of [16—18] considered the cases of vertical and horizontal surfaces embedded in a porous medium.

The studies mentioned in [15—18] deal with steady free convection. The present paper studies an unsteady free-convection boundary-layer flow near a flat vertical plate with Newtonian heating. The solution is obtained in closed form using the Laplace transform technique [19]. The literature concerning this subject can be found in the books by Slattery [20] and Carslaw and Jaeger [21].

Mathematical Analysis. We consider an unsteady free-convective flow of a viscous incompressible fluid past an impulsively started infinite vertical plate with Newtonian heating. The x^* axis is taken along the plate in the vertical upward direction and y^* axis is chosen normal to the plate. Initially, for time $t^* \le 0$, the plate and fluid are at the same temperature T_∞^* under a stationary condition. When $t^* > 0$, an impulsive motion in the vertical upward direction against gravitational field with a characteristic velocity U_c is imparted to the plate. It is assumed that the rate of heat transfer from the surface is proportional to the local surface temperature T^*. Since the plate is considered infinite in the x^* direction, all physical variables are independent of x^* and are functions of y^* and t^* only. With the use of the Boussinesq approximation, the flow is governed by the following equations:

$$\frac{\partial u^*}{\partial t^*} = \nu \frac{\partial^2 u^*}{\partial y^{*2}} + g\beta\,(T^* - T_\infty^*)\,, \tag{1}$$

$$\rho C_p \frac{\partial T^*}{\partial t^*} = k \frac{\partial^2 T^*}{\partial y^{*2}} \tag{2}$$

with the following initial and boundary conditions:

$$\begin{gathered} t^* \le 0:\ \ u^* = 0\,,\ \ T^* = T_\infty^* \quad \text{for all}\ \ y^*\,; \\ t^* > 0:\ \ u^* = U_c\,,\ \ \frac{\partial T^*}{\partial y^*} = -\frac{h}{k} T^* \ \ \text{at}\ \ y^* = 0\,, \\ u^* = 0\,,\ \ T^* = T_\infty^* \ \ \text{as}\ \ y^* \to \infty\,. \end{gathered} \tag{3}$$

We introduce the nondimensional quantities

$$u = \frac{u^*}{U_c}, \quad t = \frac{t^* U_c^2}{\nu}, \quad y = \frac{y^* U_c}{\nu}, \quad \Pr = \frac{\mu C_p}{k},$$

$$\mathrm{Gr} = \frac{\nu g \beta T_\infty^*}{U_c^3}, \quad \theta = \frac{T^* - T_\infty^*}{T_\infty^*}. \tag{4}$$

Substituting (4) into Eqs. (1)—(3) leads to the following nondimensional equations:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \mathrm{Gr}\,\theta, \tag{5}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{\Pr}\frac{\partial^2 \theta}{\partial y^2}. \tag{6}$$

The initial and boundary conditions are

$$t \le 0: \quad u = 0, \quad \theta = 0 \quad \text{for all } y;$$

$$t > 0: \quad u = 1, \quad \frac{\partial \theta}{\partial y} = -(1 + \theta) \quad \text{at } y = 0, \tag{7}$$

$$u \to 0, \quad \theta \to 0 \quad \text{as } y \to \infty.$$

The energy equation (6) is uncoupled with the momentum equation (5). One can therefore derive the temperature solution $\theta(y, t)$, whereupon the solution $u(y, t)$ can also be obtained, using the Laplace transform technique.

Solution for Pr ≠ 1. In this case the solutions look like

$$\theta(y, t) = \exp\left(-y + \frac{t}{\Pr}\right)\mathrm{erfc}\left(\frac{y\sqrt{\Pr}}{2\sqrt{t}} - \sqrt{\frac{t}{\Pr}}\right) - \mathrm{erfc}\left(\frac{y\sqrt{\Pr}}{2\sqrt{t}}\right), \tag{8}$$

$$u(y, t) = \mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right) - a\Pr\left[\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right) - \exp\left(-\frac{y}{\sqrt{\Pr}} + \frac{t}{\Pr}\right)\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} - \frac{t}{\Pr}\right)\right] -$$

$$- a\left[\left(t + \frac{y^2}{2}\right)\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right) - y\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right)\right] - a\sqrt{\Pr}\left[2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right) - y\,\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right)\right] +$$

$$+ a\Pr\left[\mathrm{erfc}\left(\frac{y\sqrt{\Pr}}{2\sqrt{t}}\right) - \exp\left(-y + \frac{t}{\Pr}\right)\mathrm{erfc}\left(\frac{y\sqrt{\Pr}}{2\sqrt{t}} - \sqrt{\frac{t}{\Pr}}\right)\right] +$$

$$+ a\left[\left(t + \frac{y^2\Pr}{2}\right)\mathrm{erfc}\left(\frac{y\sqrt{\Pr}}{2\sqrt{t}}\right) - y\sqrt{\frac{\Pr t}{\pi}}\exp\left(-\frac{y^2\Pr}{4t}\right)\right] +$$

$$+ a\sqrt{\text{Pr}}\left[2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2\text{Pr}}{4t}\right) - y\sqrt{\text{Pr}}\,\text{erfc}\left(\frac{y\sqrt{\text{Pr}}}{2\sqrt{t}}\right)\right], \tag{9}$$

where $a = \frac{\text{Gr}}{\text{Pr} - 1}$, erfc (x) is the complementary error function defined as

$$\text{erfc}\,(x) = 1 - \text{erf}\,(x)\,, \quad \text{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\,(-\eta^2)\,d\eta\,.$$

Solution at Pr = 1. We see that the solution for the velocity given by Eq. (9) is not valid for fluids at a Prandtl number equal to unity. As the Prandtl number is a measure of the relative importance of the viscosity and thermal conductivity of the fluid, the case Pr = 1 corresponds to those fluids whose momentum and thermal boundary layer thicknesses are of the same order of magnitude. It may be noted that the solution for the temperature $\theta(y, t)$ follows from Eq. (8) when Pr = 1. However, in the case of $u(y, t)$ the solution has to be re-derived starting from Eqs. (5) and (6). The solution obtained is

$$u\,(y, t) = \text{erfc}\left(\frac{y}{2\sqrt{t}}\right) + \frac{\text{Gr}\,y}{2}\left[-2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{y^2}{4t}\right) + (y - 1)\,\text{erfc}\left(\frac{y}{2\sqrt{t}}\right) + \exp\,(-y + t)\,\text{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{t}\right)\right]. \tag{10}$$

Skin friction. Knowing the velocity field, we now obtain the skin friction which is given as

$$\tau^* = -\mu\left(\frac{\partial u^*}{\partial y^*}\right)_{y^*=0} \tag{11}$$

and by virtue of Eq. (4) this equation is reduced to

$$\tau = \frac{\tau^*}{\rho U_c^2} = -\left.\frac{du}{dy}\right|_{y=0}\,. \tag{12}$$

For Pr ≠ 1

$$\tau = \frac{1}{\sqrt{\pi\,t}} - \frac{\text{Gr}\sqrt{\text{Pr}}}{\sqrt{\text{Pr}} + 1}\left[\exp\left(\frac{t}{\text{Pr}}\right)\left(1 + \text{erf}\left(\sqrt{\frac{t}{\text{Pr}}}\right)\right)\right] + \frac{2\text{Gr}}{\sqrt{\text{Pr}} + 1}\sqrt{\frac{t}{\pi}} + \frac{\text{Gr}\sqrt{\text{Pr}}}{\sqrt{\text{Pr}} + 1}\,, \tag{13}$$

at Pr = 1

$$\tau = \frac{1}{\sqrt{\pi\,t}} + \frac{\text{Gr}}{2}\left[2\sqrt{\frac{t}{\pi}} + 1 - \exp\,(t)\,(1 + \text{erf}\,(\sqrt{t}))\right]. \tag{14}$$

Nusselt number. Another purpose of this study is to understand the effects of t and Pr on the Nusselt number. In nondimensional form, the heat transfer coefficient is given as

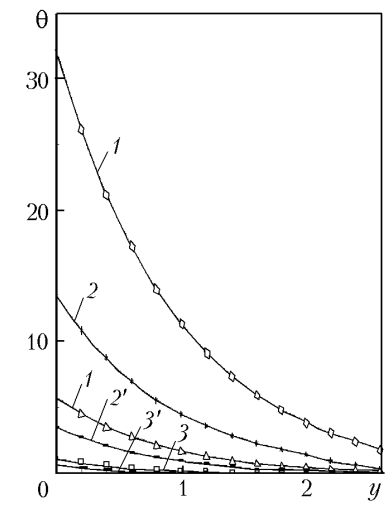
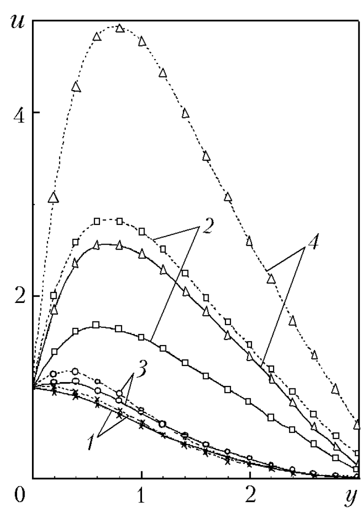


Fig. 1. Temperature profiles at Pr = 0.71 (1 and 1′), 1 (2 and 2′), and 7 (3 and 3′): 1, 2, 3) t = 2; 1′, 2′, and 3′) t = 0.9.



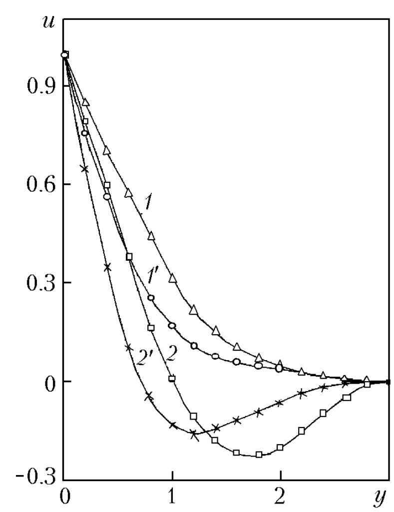
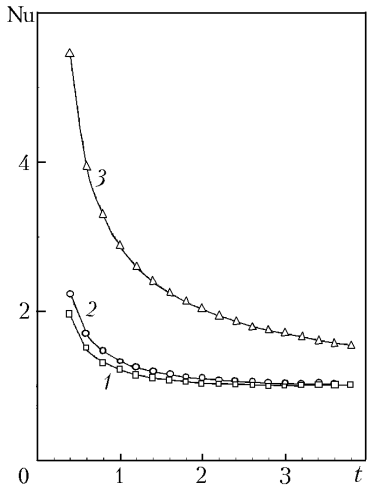


Fig 2. Velocity profiles at Pr = 0.71 (dotted curves) and 1 (solid curves): 1) Gr = 0.5 and t = 0.9; 2) 0.5 and 2; 3) 1 and 0.9; 4) 1 and 2.

Fig 3. Velocity profiles at Pr = 7 and Gr = 0.5 (1 and 1′) and Gr = 1 (2 and 2′): 1, 2) t = 2; 1′ 2′) t = 0.9.

$$\mathrm{Nu} = \frac{1}{\theta(0)} + 1 = \frac{1}{\exp\left(\frac{t}{\mathrm{Pr}}\right)\left(1 + \mathrm{erf}\left(\sqrt{\frac{t}{\mathrm{Pr}}}\right)\right) - 1} + 1 .$$

Discussion and Conclusions. In order to discuss the effect of various physical parameters on the velocity field, thermal boundary layer, skin friction (shear stress), and the coefficient of heat transfer on the wall, the numerical calculation of the solutions obtained in the preceding section was carried out. The results are presented in Figs. 1—5. The values of the Prandtl number are taken equal to 0.71, 1.0, and 7.0, which correspond to air, electrolyte solution, and water, respectively.



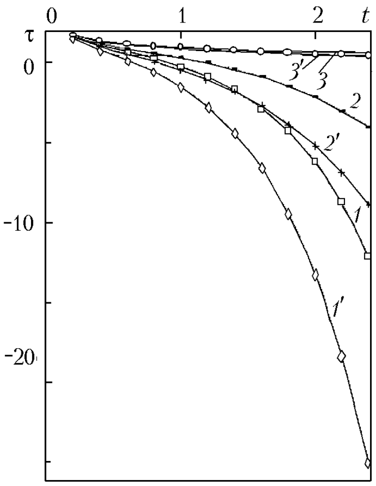


Fig. 4. The Nusselt number as a function of time at Pr = 0.71 (1), 1 (2), and 7 (3).

Fig. 5. Skin friction as a function of time for Pr = 0.71 (1 and 1′), 1 (2 and 2′), and 7 (3 and 3′): 1, 2, 3) Gr = 0.5; 1′, 2′, 3′) Gr = 1.

The effect of the Prandtl number on the temperature profiles $\theta(y, t)$ may be analyzed from Fig. 1. It is inferred that the thickness of the thermal boundary layer is the greatest for air (Pr = 0.71), when the temperature distribution across the thermal boundary layer is less uniform as compared to water (Pr = 7.0) and electrolyte solution (Pr = 1.0). From this figure we notice that an increase in the Prandtl number results in a decrease of temperature. The reason is that smaller values of the Prandtl number are equivalent to increasing thermal conductivity, and therefore heat is capable of diffusing away from the heated surface more rapidly than at higher values of Pr. Thus the temperature falls more rapidly for water than for air and electrolyte solution. The temperature maximum occurs in the vicinity of the plate and then the temperature asymptotically approaches zero in the free-stream region. Furthermore it is found that the thermal boundary layer thickens with time.

Figure 2 presents the velocity profiles $u(y, t)$ for the electrolyte solution and air inside the boundary layer for two values of the Grashof number and time, while Fig. 3 depicts these profiles for water. Figure 2 shows that the fluid velocity increases with the Grashof number. Physically, this is possible because with increase in the Grashof number the contribution from the buoyancy near the plate becomes significant, and hence a rise in the velocity in this range is observed. For higher values of Gr, the fluid velocity overshoots the plate velocity in the regions close to the boundary, and this overshooting is more pronounced for fluids with lower Prandtl number. The thickness of the momentum boundary layer is also greater for these fluids. The reason for such a behavior is the fact that an increase in the Prandtl number is due to an increase in the fluid viscosity, which makes the fluid thick, thus decreasing its velocity. In the case of water (see Fig. 3), the velocity decreases from the plate velocity to the zero free-stream value. From this figure it is also seen that the water velocity decreases with increasing Gr. Further, Figs. 2 and 3 show that the thickness of the momentum boundary layer increases with time.

Figure 4 displays the Nusselt number Nu vs. t. This figure shows that this number increases with Pr and decreases with increasing t. It should be noted that heat is transferred from the surface to the medium.

The influence of the buoyancy parameter Gr and the Prandtl number on the time dependence of the skin friction τ is presented in Fig. 5. It is seen that the skin friction falls with increasing Grashof number and time. However, τ increases with Pr.

NOTATION

C_p, specific heat at constant pressure; g, acceleration due to gravity; Gr, Grashof number; h, heat transfer coefficient; k, thermal conductivity; Nu, Nusselt number; Pr, Prandtl number; T^*, temperature; T_∞^*, ambient temperature;

t, nondimensional time; u, nondimensional velocity along x direction; x and y, nondimensional Cartesian coordinates along the plate and normal to it; β, coefficient of volumetric expansion; θ, nondimensional temperature; μ and ν, viscosity and kinematic viscosity; ρ, fluid density; τ, nondimensional skin friction. Superscripts: *, dimensional quantities.

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